Semi-Markov model for market microstructure and HFT

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6th General AMaMeF and Banach Center Conference
10-15 June 2013

Joint work with Huyên PHAM
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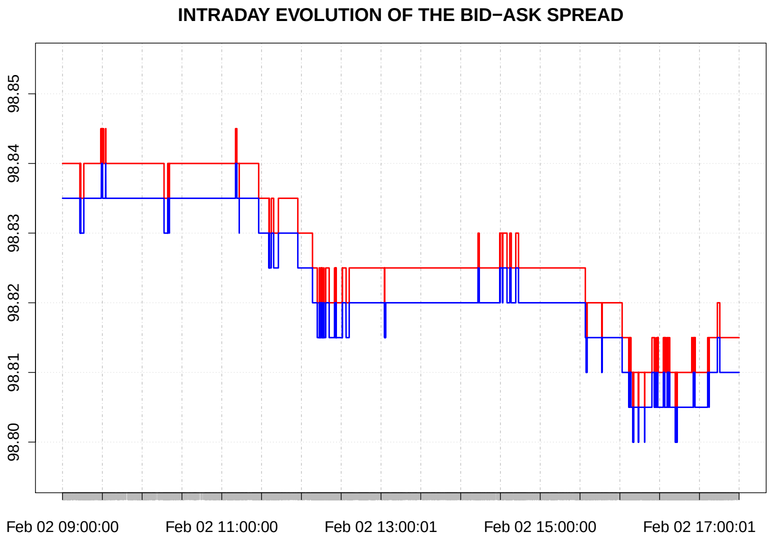

INTRADAY EVOLUTION OF THE BID−ASK SPREAD
98.85
98.84
98.83
98.82
98.81
98.80
Feb 02 09:00:00
Feb 02 11:00:00
Feb 02 13:00:01
Feb 02 15:00:00
Feb 02 17:00:01

What price do we want to model?

Several prices in the limit order book:

quote prices: best ask price, best bid price, **mid** price (offered liquidity)

trade prices: last transaction price, vwap (consummed liquidity)

We have chosen to model:

- the mid price
- of liquid assets where the bid-ask spread is constantly one tick
- all the quotes prices can be derived from the mid one

Stylized fact

1 - Microscopic mean reversion

Short-term returns are usually anticorrelated.

2 - Clustering

Independently from the seasonal patterns, market alternates period of high and low activity.

3 - Point process with diffusive limit

The price process is piecewise constant, and so not diffusive. Anyway, at large scales, its behavior can be approximated by a Brownian motion.

4 - Explosion of the realized volatility

The volatility estimation depends on the sample frequency: the higher is the frequency the biggest is the realized volatility.

Tracability requirements

Estimation

Easy, fast and non parametric

Simulation

Easy, fast and exact

Markov property

Markov embedding with few state variables to use and solve numerically HJB equations

Model-free description of asset mid-price

Constant bid-ask spread = 1 tick = 2δ

- The **timestamps** $(T_k)_k$ of its jump times modeling of volatility clustering
- The **marks** $(J_k)_k$ valued in $2\delta\mathbb{Z}\setminus\{0\}$, representing the price increment at T_k: modeling of the microstructure noise via mean-reversion of price increments

Model-free dynamic of the price

$$P_t = P_0 + 2\delta \sum_{k:\, T_k \le t} J_k$$

Jump side modeling
Case $|J_k| = 1$

- J_k valued in $\{+1, -1\}$: side of the jump (upwards or downwards)

$$J_k = J_{k-1} \cdot B_k \Rightarrow \mathbb{E}[J_k] = 0 \text{ under stationary prob}$$

- $(B_k)_k$ i.i.d. with law:

$$\mathbb{P}[B_k = \pm 1] = \frac{1 \pm \alpha}{2}, \quad \alpha \in [-1, 1)$$

- $(J_k)_k$ irreducible Markov chain with symmetric transition matrix:

$$Q_\alpha = \begin{pmatrix} \frac{1+\alpha}{2} & \frac{1-\alpha}{2} \\ \frac{1-\alpha}{2} & \frac{1+\alpha}{2} \end{pmatrix}$$

Mean reversion

1 - Microscopic mean reversion: $\alpha < 0$

- Under the stationary probability of $(J_k)_k$, we have:

$$\alpha = Cor(J_k, J_{k-1})$$

- Estimation of α:

$$\hat{\alpha}_n = \frac{1}{n}\sum_{k=1}^{n} J_k J_{k-1}$$

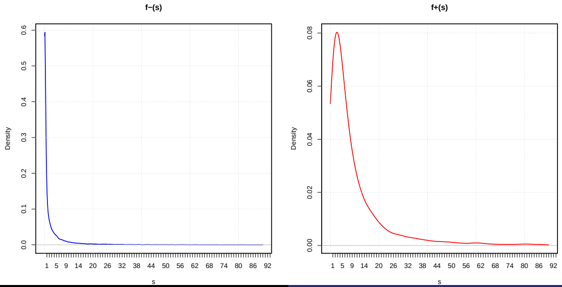
- $\alpha \approx -87.5\%$ (Euribor, 2010, 10h-14h): strong mean reversion of price returns

2 - Clustering

Renewal law

Conditionally on $\{J_k J_{k-1} = \pm 1\}$, the sequence of inter-arrival jump times $\{S_k = T_k - T_{k-1}\}$ is i.i.d. with distribution function $F_\pm$ and density $f_\pm$:

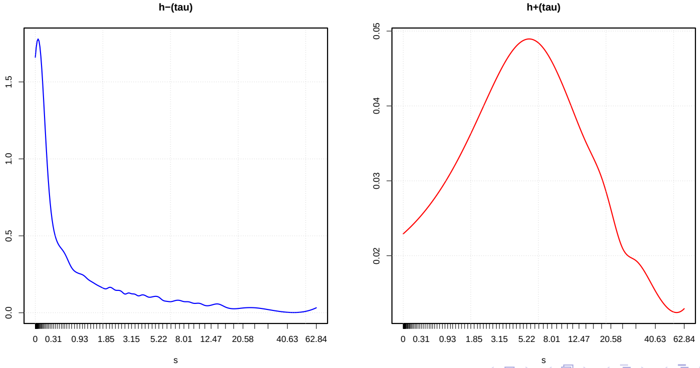
$$F_\pm(t) \quad = \quad \mathbb{P}\left[S_k \leq t \mid J_k \cdot J_{k-1} = \pm 1\right].$$



Non parametric estimation of the jump intensity

The price changes with intensity:

$$\hat{h}_{\pm}(s) = \frac{1 \pm \alpha}{2} \frac{f_{\pm}(s)}{1 - \frac{1+\alpha}{2} F_{+}(s) - \frac{1-\alpha}{2} F_{-}(s)}$$



3 - Point process with diffusive limit

$$P_t^{(T)} = \frac{P_{tT}}{\sqrt{T}}, \quad t \in [0,1].$$

Diffusive behaviour

$$\lim_{T \to \infty} P^{(T)} \overset{(d)}{=} \sigma_\infty W,$$

where W is a Brownian motion, and:

$$\sigma_\infty^2 = function(F, \alpha)$$

Simulated price

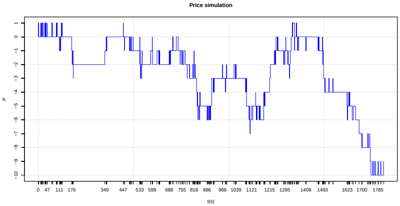


Figure: 30 minutes simulation

Figure: 1 day simulation

4 - Explosition of the realized volatility

In the special case when $(T_k)_k$ and $(J_k)_k$ are independent

Mean Signature Plot

$$\bar{V}(\tau) := \frac{1}{\tau}\mathbb{E}\left[(P_\tau - P_0)^2\right] \; = \; \sigma_\infty^2 + \phi(\alpha, \tau)$$

- ϕ is semi-explicit
- ϕ is finite
- for $\alpha < 0$, ϕ is decreasing in τ
- $\phi(\tau, \alpha)$ goes to 0 when $\tau \to \infty$

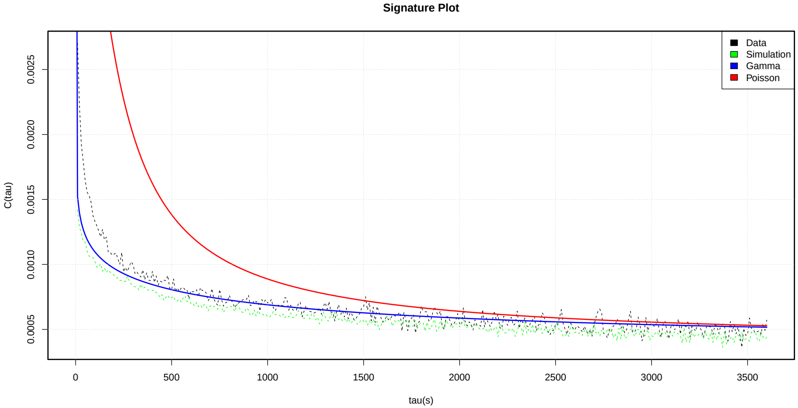


Figure: Mean signature plot for $\alpha < 0$

Trading issue

Market making

Agent submitting limit orders on both sides of the LOB:

- limit buy order at the best bid price
- limit sell order at the best ask price

with the aim to gain the spread.

- need to model the market order flow, i.e. the counterpart trade of the limit order
- need to model the agent execution

Market trades

Market order flow $\Leftrightarrow$ marked point process $(\theta_k, Z_k)_k$

- θ_k: arrival time of the market order $\leftrightarrow$ M_t counting process
- Z_k: valued in $\{-1, +1\}$: side of the trade
    - $Z_k = -1$: trade at the best BID price (market sell order)
    - $Z_k = +1$: trade at the best ASK price (market buy order)

index n	θ_k	best ask	best bid	traded price	Z_k
1	9:00:01.123	98.47	98.46	98.47	$+1$
2	9:00:02.517	98.47	98.46	98.46	-1
3	9:00:02.985	98.48	98.47	98.47	-1

Dependence modeling between market order flow and price in LOB

Trade timestamp modeling

The trade counting process M_t

The counting process (M_t) of the market order timestamps $(\theta_k)_k$ is a Cox process with conditional intensity $\lambda(S_t)$,where:

$$S_t = \text{ time elapsed since the last price change}$$

- Parametric examples (positive parameters):

$$\lambda^{exp}(s) = \lambda_0 + \lambda_1 s^r e^{-ks}$$
$$\lambda^{pow}(s) = \lambda_0 + \frac{\lambda_1 s^r}{1 + s^k}$$

- Estimation by MLE minimizing

$$\sum_k \int_0^{S_k} \lambda(s)\, ds - \sum_j \ln[\lambda(S(\theta_j -))]$$

Strong and weak side of LOB

- We call strong side (+) of the LOB, the side in the same direction than the last jump, e.g. best ask when price jumped upwards.
- We call weak side (−) of the LOB, the side in the opposite direction than the last jump, e.g. best bid when price jumped upwards.

Empirical fact

We observe that trades (market orders) arrive mostly on the weak side.

Mean reversion

Trade side modelling

For an incoming trade, the probability that the trade is exchanged on the strong(+)/weak(-) side is:

$$\frac{1 \pm \rho}{2}, \quad \rho \in [-1, 1]$$

- $\rho = 0$: market order flow arrive independently at best bid and best ask (usual assumption in the existing literature)
- $\rho > 0$: market orders arrive more often in the strong side of the LOB
- $\rho < 0$: market orders arrive more often in the weak side of the LOB

1. $\hat{\rho}_n = \frac{1}{n} \sum_{k=1}^{n} Z_k \, I_{\theta_k^-} \simeq -50\%$: $\approx 3/4$ trades on the weak side
2. ρ has an impact on the stragegy performance

Market making strategy

Agent control

Predictable process $(\ell_t^+, \ell_t^-)_t \in \{0, 1\}$

- $\ell_t^+ = 1$: limit order of size L on the strong side: $+I_{t^-}$
- $\ell_t^- = 1$: limit order of size L on the weak side: $-I_{t^-}$

Agent execution

If the agent is placed, she can be executed:

- entirely, if the price jumps over her limit order
- randomly if a trade arrives

Market making optimization

Value function

- S_t the time past since the last price change
- I_t the last direction taken by the price
- X_t the cash process
- Y_t the inventory process

$$v(t,s,p,i,x,y) = \sup_{(\ell^+,\ell^-)} \mathbb{E}\left[PNL_T - CLOSE(Y_T) - \eta \cdot RISK_{t,T}\right]$$

where $\eta \geq 0$ is the agent risk aversion and:

$$\begin{aligned} PNL_t &= X_t + Y_t \cdot P_t \quad \text{(ptf valued at the mid price)} \\ CLOSE(y) &= -(\delta + \epsilon) \cdot |y| \quad \text{(closure market order)} \\ RISK_{t,T} &= \int_t^T Y_u^2 \cdot d[P]_u \quad \text{(no inventory imbalance)} \end{aligned}$$

Variable reduction

Theorem

The value function is given by:

$$v(t,s,p,i,x,y) \quad = \quad x + yp + \omega_{yi}(t,s)$$

where $\omega_q(t,s) = \omega(t,s,q)$ is the unique viscosity solution to:

$$\begin{aligned} &[\partial_t + \partial_s - \hat{\kappa}(s)]\,\omega + \sigma^2(s)\cdot[\alpha q - \eta q^2] \\ + &\max_{\ell\in\{0,1\},q-\ell L\in\mathbb{Y}} \mathcal{L}_+^\ell\,\omega + \max_{\ell\in\{0,1\},q+\ell L\in\mathbb{Y}} \mathcal{L}_-^\ell\,\omega = 0 \\ &\omega_q(T,s) = -|q|\,(\delta+\epsilon) \end{aligned}$$

in $[0,T]\times\mathbb{R}_+\times\mathbb{Y}$.

The effect of ρ (adverse selection)

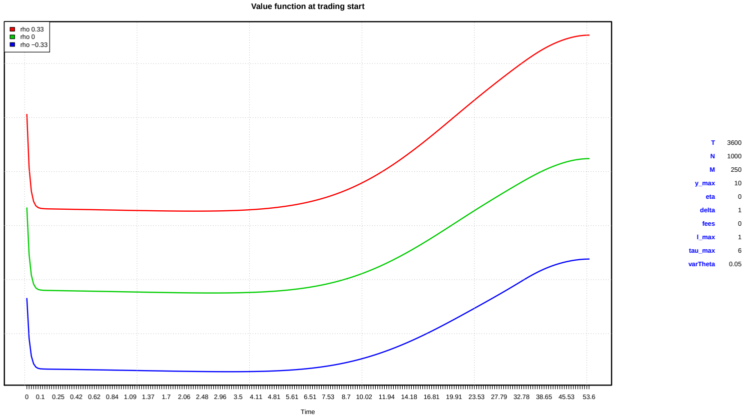


Figure: Adverse selection: value function increasing in ρ

Optimal policy shape

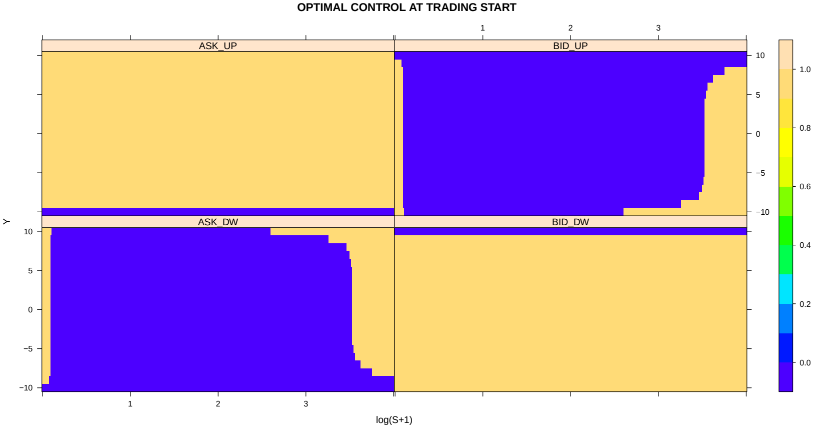


Figure: Always play on the strong side!